# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL NOTE 3700

## ON SUBSONIC FLOW PAST A PARABOLOID OF REVOLUTION

By Carl Kaplan

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
February 1957

1E

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3700

# ON SUBSONIC FLOW PAST A PARABOLOID OF REVOLUTION

By Carl Kaplan

## SUMMARY

The Janzen-Rayleigh method of expansion in powers of the stream Mach number $M_\infty$ is utilized for the calculation of the velocity potential for steady subsonic flow past a paraboloid of revolution. Only the first two terms of this expansion are calculated, the first term being the incompressible expression for the velocity potential and the second being the term in $M_\infty^2$. A closed expression is obtained for the second term in the form of a double infinite integral which contains Bessel functions under the integral signs. The methods of evaluating such integrals are not very numerous. Unfortunately, the present integral does not yield to any of them. No attempt is made in the present paper to evaluate numerically this double infinite integral. Expressions for the fluid velocity are given in the form of correction factors by which the corresponding expressions for incompressible flow are multiplied in order to take into account the effect of compressibility.

## INTRODUCTION

The Janzen-Rayleigh method for the calculation of subsonic flow past an obstacle, by expansion of the velocity potential in powers of the undisturbed stream Mach number, has been extensively applied to two-dimensional problems. A considerable void, however, exists in the literature insofar as applications of this method to three-dimensional axisymmetric problems are concerned. Except for the case of flow past a sphere and sporadic efforts to treat the next simplest case of axisymmetric flow past a prolate spheroid, little has been accomplished in this area of subsonic compressible-flow theory. Clearly, then, future work utilizing the Janzen-Rayleigh method can be expended profitably on axisymmetric-flow problems. A first step in this direction has been taken by A. L. Longhorn who recently, at the suggestion of M. J. Lighthill, reconsidered the problem of subsonic flow past a prolate spheroid. (See ref. 1.) The problem treated in the present paper was begun sometime before the author became aware of the results of Longhorn. Fortunately, the two problems complement one another in the sense that the one treated by Longhorn is limited to closed bluff bodies, whereas the present one is concerned with a family of semi-infinite elongated bodies with round noses.

The choice of the paraboloid of revolution as the solid body was made for several reasons. First, the paraboloid being a semi-infinite body and, in fact, a limiting case of the prolate spheroid, it was thought probable that the analysis might involve functions of a more elementary nature than for the case of a closed body. Second, the fluid speed at the surface rises monotonically from zero at the stagnation point to the undisturbed-stream value at infinity. Therefore, the critical value of the stream Mach number is unity, and hence there can be no transonic influence in the entire subsonic range. Finally, the Janzen-Rayleigh method being a thoroughly reliable one, the present investigation should provide useful information with regard to the question of the accuracy of the small-disturbance method for the calculation of compressible flow past slender bodies in the neighborhood of the stagnation point.

## ANALYSIS

The Janzen-Rayleigh method has often been described in the literature. In the present paper, therefore, only those equations necessary for the formulation of the problem are used. The problem to be considered is the subsonic flow past a paraboloid of revolution fixed in a uniform stream of velocity $U$ in the negative direction of the axis of symmetry. The nature of axisymmetrical flow is such that the motion is the same in every (meridian) plane through the axis of symmetry. The position of a point in a meridian plane may be fixed by rectangular Cartesian coordinates $x,y$ with the origin at the focus of the parabolic meridian profile. (See fig. 1.) With the radius of curvature at the nose as the unit of length, the equation of the meridian profile becomes

$$y^2 = -2\left(x - \frac{1}{2}\right) \tag{1}$$

For steady subsonic axisymmetric flows, the equation satisfied by the velocity potential $\phi$ is the equation of continuity

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_\infty} y \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\rho_\infty} y \frac{\partial \phi}{\partial y}\right) = 0 \tag{2}$$

where

$$\frac{\rho}{\rho_\infty} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2(q^2 - 1)\right]^{\frac{1}{\gamma-1}} \tag{3}$$

and

| | |
|---|---|
| $q$ | nondimensional speed of fluid with $U$ as unit of velocity |
| $\rho$ | density of fluid moving with speed $q$ |
| $\rho_\infty$ | density of fluid moving with undisturbed speed $U$ |
| $M_\infty$ | Mach number of undisturbed stream, $\frac{U}{c_\infty}$ |
| $c_\infty$ | speed of sound in undisturbed stream |
| $\gamma$ | ratio of specific heats at constant pressure and constant volume |

If, now, a series expansion for $\phi$ in powers of ${M_\infty}^2$ is assumed, then

$$\phi = \sum_{n=0}^{\infty} \phi_n {M_\infty}^{2n}$$

where the first term $\phi_o$ is the velocity potential in incompressible flow. In this paper, only $\phi_1$ is calculated so that the form of $\phi$ to be found is

$$\phi = \phi_o + {M_\infty}^2 \phi_1 \tag{4}$$

Thus, this expression for $\phi$ is substituted into equation (2), and equation (3) is utilized with

$$q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2$$

Then, when the coefficients of the zeroeth and first powers of ${M_\infty}^2$ are equated to zero, the following pair of equations is obtained:

$$\frac{\partial}{\partial x}\left(y\,\frac{\partial\phi_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\,\frac{\partial\phi_o}{\partial y}\right) = 0 \tag{5}$$

and

$$\frac{\partial}{\partial x}\left(y\,\frac{\partial\phi_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\,\frac{\partial\phi_1}{\partial y}\right) = \frac{1}{2}\,\frac{\partial}{\partial x}\left[y\left({q_o}^2 - 1\right)\frac{\partial\phi_o}{\partial x}\right] + \frac{1}{2}\,\frac{\partial}{\partial y}\left[y\left({q_o}^2 - 1\right)\frac{\partial\phi_o}{\partial y}\right] \tag{6}$$

where $q_o$ is the local velocity in incompressible flow.

The velocity potential for incompressible flow past the paraboloid of revolution defined by equation (1) is readily found to be

$$\phi_o = -x + \frac{1}{2}\log\left(x + \sqrt{x^2 + y^2}\right) \tag{7}$$

where $\phi_o$ is nondimensional with the undisturbed velocity $U$ as unit of velocity and the radius of curvature at the nose as unit of length. Equation (7) satisfies equation (5) and the boundary conditions of vanishing normal velocity at the surface and of vanishing disturbance velocity infinitely far from the paraboloid. The expression for the magnitude of the incompressible fluid velocity is

$$q_o^2 = 1 - \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{2y^2}\left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right) \tag{8}$$

At this point it is convenient to introduce a new set of independent variables. Ideally, the appropriate coordinate system for the present problem is a parabolic one which defines mutually orthogonal families of confocal parabolas in a meridian plane. Thus, the conformal transformation

$$z = x + iy = (\xi + i\eta)^2 = \zeta^2$$

gives

$$\left.\begin{aligned} x &= \xi^2 - \eta^2 \\ y &= 2\xi\eta \end{aligned}\right\} \tag{9}$$

The elimination of the variable $\eta$ in equations (9) yields

$$x - \xi^2 = -\frac{y^2}{4\xi^2}$$

Then the surfaces $\xi = \text{Constant} = \xi_o$ are confocal paraboloids of revolution with the focus at the origin and the radius of curvature at the nose equal to $2\xi_o^2$. With this radius of curvature as unit of length, equations (9) are nondimensional and the solid boundary is given by $\xi_o = \frac{1}{\sqrt{2}}$ or equation (1).

The equations corresponding to equations (5) and (6) are

$$\frac{\partial}{\partial\alpha}\left(\alpha \frac{\partial\phi_o}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\beta \frac{\partial\phi_o}{\partial\beta}\right) = 0 \tag{10}$$

and

$$\frac{\partial}{\partial\alpha}\left(\alpha \frac{\partial\phi_1}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\beta \frac{\partial\phi_1}{\partial\beta}\right) = \frac{1}{2} \frac{\partial}{\partial\alpha}\left[\alpha \left(q_o^2 - 1\right)\frac{\partial\phi_o}{\partial\alpha}\right] + \frac{1}{2} \frac{\partial}{\partial\beta}\left[\beta \left(q_o^2 - 1\right)\frac{\partial\phi_o}{\partial\beta}\right] \tag{11}$$

where

$$\xi^2 = \alpha \quad \text{and} \quad \eta^2 = \beta$$

Also, equations (7) and (8) become, respectively,

$$\phi_o = -\alpha + \beta + \frac{1}{2} \log 2\alpha \tag{12}$$

and

$$q_o^2 - 1 = -\frac{1}{\alpha + \beta} + \frac{1}{4\alpha(\alpha + \beta)} \tag{13}$$

Then, equation (11) for the velocity potential $\phi_1$ takes the form,

$$\frac{\partial}{\partial\alpha}\left(\alpha \frac{\partial\phi_1}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\beta \frac{\partial\phi_1}{\partial\beta}\right) = -\frac{1}{2} \frac{\alpha - \beta}{(\alpha + \beta)^2} + \frac{1}{2} \frac{1}{(\alpha + \beta)^2} - \frac{1}{16\alpha^2(\alpha + \beta)} - \frac{1}{16\alpha(\alpha + \beta)^2} \tag{14}$$

The complementary solution of this partial-differential equation is easily found by assuming a solution in product form:

$$\phi_1 = A(\alpha)B(\beta)$$

Then the homogeneous form of equation (14) becomes

$$\frac{1}{A} \frac{d}{d\alpha}\left(\alpha \frac{dA}{d\alpha}\right) + \frac{1}{B} \frac{d}{d\beta}\left(\beta \frac{dB}{d\beta}\right) = 0$$

or

$$\left.\begin{aligned} \frac{d}{d\alpha}\left(\alpha \frac{dA}{d\alpha}\right) - \frac{\lambda^2}{4} A = 0 \\ \frac{d}{d\beta}\left(\beta \frac{dB}{d\beta}\right) + \frac{\lambda^2}{4} B = 0 \end{aligned}\right\} \tag{15}$$

where $\lambda$ is an arbitrary real constant. Equations (15) are equivalent forms of Bessel's differential equation for cylinder functions of order zero. Therefore,

$$A = C_0(i\lambda\sqrt{\alpha})$$

and

$$B = C_0(\lambda\sqrt{\beta})$$

where $C_0$ represents any cylinder function of order zero.

The nature of the present problem is such that the cylinder function involving the variable $\alpha$ shall behave well for $\alpha \longrightarrow \infty$ but not necessarily for $\alpha \longrightarrow 0$ since $\alpha = \frac{1}{2}$ at the solid boundary and never takes on values less than that. On the other hand, the cylinder function involving $\beta$ must be of such type that it behaves well for the entire range $0 \gtreqless \beta \gtreqless \infty$. Clearly then,

$$A = K_0(\lambda\sqrt{\alpha})$$

and

$$B = J_0(\lambda\sqrt{\beta})$$

where $K_0$ is the modified Bessel function of the second kind and zero order and $J_0$ is the Bessel function of the first kind and zero order. The graph of $K_0$ resembles a rectangular hyperbola in the first quadrant and that of $J_0$, a damped cosine wave. The general complementary solution $\phi_{1c}$ of equation (14) can then be written as

$$\phi_{1c} = \sum_{n=1}^{\infty} c_n K_0\left(\lambda_n\sqrt{\alpha}\right) J_0\left(\lambda_n\sqrt{\beta}\right) \tag{16}$$

where the values of $c_n$ are arbitrary constants to be determined by means of the boundary condition that the fluid-velocity component normal to the surface of the paraboloid shall vanish.

A suitable particular integral of equation (14) is now sought. Unfortunately, the right-hand side of that equation cannot be separated into a sum of products of functions of $\alpha$ alone and of $\beta$ alone; hence, the task of finding a particular integral directly from equation (14) is practically futile. However, by temporary use of polar coordinates as independent variables the desired separation of variables can be achieved. Thus,

$$\left.\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}\right\} \tag{17}$$

where the radius vector $r$ is nondimensional with the radius of curvature at the nose of the paraboloid as unit of length and where the angle $\theta$ is measured positive counterclockwise. (See fig. 1.)

The polar equation of the parabolic meridian profile is

$$r = \frac{1}{1 + \cos \theta} \tag{18}$$

and the expressions for the incompressible velocity potential and fluid speed given by equations (7) and (8), respectively, are

$$\left.\begin{aligned} \phi_o &= -r \cos \theta + \frac{1}{2} \log r(1 + \cos \theta) \\ {q_o}^2 &= 1 - \frac{1}{r} + \frac{1}{2r^2(1 + \cos \theta)} \end{aligned}\right\} \tag{19}$$

Thus, at the upper surface of the paraboloid of revolution, the expression for the incompressible fluid speed is simply

$$q_o = \sin \frac{1}{2} \theta$$

It is interesting to note the curious fact that this expression for the velocity at the surface of a paraboloid of revolution is precisely the same as that for the velocity at the surface of a two-dimensional parabolic cylinder. The explanation lies in Munk's rule which states that the surface velocity on any ellipsoid immersed in a uniform flow along a principal axis is the projection of the maximum velocity (in the present case, unity) on the tangent plane to the surface. (See ref. 2.) This is obviously the same for both a parabolic cylinder and a paraboloid of revolution in view of the fact that Munk's rule includes the two-dimensional case.

When equations (17) and (19) are utilized, differential equation (6) for the velocity potential $\phi_1$ takes the following form:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_1}{\partial r}\right) + \frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial \phi_1}{\partial \mu}\right] = -\frac{1}{2}\mu + \frac{1}{2r} - \frac{1}{8r^2}\left[\frac{1}{1 + \mu} + \frac{2}{(1 + \mu)^2}\right] \tag{20}$$

where $\mu = \cos \theta$.

Note that the desired separation of variables has been achieved on the right-hand side of equation (20). The task of finding a particular integral then becomes a routine problem. The following particular integral $\phi_{1p}$ of equation (20) has been constructed with the point in mind that it be well behaved everywhere at the surface of the boundary and in the field of flow:

$$\begin{aligned}\phi_{1p} = \frac{1}{4}\mu - \frac{1}{8r}\log r(1 + \mu)^3 + \frac{1}{8r^2}\Bigg[&1 - \log 2 - \frac{1}{1 + \mu} - \\ &\frac{1}{4}\mu \log^2(1 + \mu) + \left(1 + \frac{1}{2}\mu \log 2\right)\log(1 + \mu) - \\ &\frac{1}{2}\mu \int_0^{\mu} \frac{\log(1 + \mu) - \log 2}{1 - \mu}\, d\mu\Bigg]\end{aligned} \tag{21}$$

This particular integral may be expressed in terms of parabolic coordinates by means of the following equations of transformation:

$$\left.\begin{aligned} r &= \alpha + \beta \\ \mu &= \frac{\alpha - \beta}{\alpha + \beta}\end{aligned}\right\} \tag{22}$$

The general solution for the velocity potential $\phi_1$ then becomes

$$\begin{aligned}\phi_1 = \frac{1}{4}\frac{\alpha - \beta}{\alpha + \beta} - \frac{1}{8}\frac{1}{\alpha + \beta}\log\frac{8\alpha^3}{(\alpha + \beta)^2} + \frac{1}{8(\alpha + \beta)^2}\Bigg[&\frac{1}{2} - \log 2 - \frac{\beta}{2\alpha} - \\ &\frac{1}{4}\frac{\alpha - \beta}{\alpha + \beta}\log^2\frac{2\alpha}{\alpha + \beta} + \left(1 + \frac{1}{2}\frac{\alpha - \beta}{\alpha + \beta}\log 2\right)\log\frac{2\alpha}{\alpha + \beta} - \\ &\frac{1}{2}\frac{\alpha - \beta}{\alpha + \beta}\int_0^{\frac{\alpha-\beta}{\alpha+\beta}} \frac{\log(1 + \mu) - \log 2}{1 - \mu}\, d\mu\Bigg] + \sum_{n=1}^{\infty} c_n K_o\left(\lambda_n\sqrt{\alpha}\right)J_o\left(\lambda_n\sqrt{\beta}\right)\end{aligned} \tag{23}$$

where the arbitrary constants $c_n$ are determined by means of the boundary condition that the normal component of the fluid velocity at the surface of the paraboloid shall vanish. In addition, there is the requirement that the disturbance velocity shall vanish at infinity for points not near the paraboloid. Now, $\phi_o$, being the solution for incompressible flow, satisfies these boundary conditions, and hence $\phi_1$ must also separately satisfy them. An examination of equation (23) shows that the disturbance velocity vanishes at infinity. The normal boundary condition $\left(\frac{\partial\phi_1}{\partial\alpha}\right)_{\alpha=\frac{1}{2}} = 0$ then yields the following equation for the determination of the arbitrary constants:

$$\sum_{n=1}^{\infty} d_n J_o\left(\lambda_n\sqrt{\beta}\right) = \frac{1}{(1 + 2\beta)^3}\left\{\frac{3}{2}(1 - 2\beta)\left[-1 + \log(1 + 2\beta)\right] + (1 + 2\beta)\log 2\right\} + \frac{1 - 4\beta}{(1 + 2\beta)^4}\left[-\frac{1}{2}\log^2 2 + \frac{1}{2}\log^2(1 + 2\beta) + I(\beta)\right]$$

$$\equiv F(\beta) \tag{24}$$

where, for convenience of expression,

$$d_n = \frac{\lambda_n}{\sqrt{2}} K_1\left(\frac{\lambda_n}{\sqrt{2}}\right)c_n$$

and

$$I(\beta) = \int_0^{\frac{1-2\beta}{1+2\beta}} \frac{\log(1 + \mu) - \log 2}{1 - \mu} d\mu$$

Now, the range of the variable $\beta$ along the upper surface of the solid boundary extends from 0 to $\infty$. Therefore, the determination of the complementary function $\phi_{1c}$ must proceed along lines corresponding to the passage from a Fourier series to a Fourier integral. Thus, multiplying both sides of equation (24) by $\sqrt{\beta}J_o\left(\lambda_m\sqrt{\beta}\right)d\sqrt{\beta}$ and integrating from 0 to an arbitrary upper limit b give

$$d_n = \frac{2}{\left[bJ_1(b\lambda_n)\right]^2}\int_0^b F(\beta)\sqrt{\beta}J_o(\lambda_n\sqrt{\beta})\,d\sqrt{\beta} \tag{25}$$

where $\lambda_1$, $\lambda_2$, . . . are different values of $\lambda$ for which $J_o(b\lambda_n) = 0$. In equation (25), use has been made of the well-known orthogonality condition

$$\int_0^b \sqrt{\beta}J_o\left(\lambda_m\sqrt{\beta}\right)J_o\left(\lambda_n\sqrt{\beta}\right)d\sqrt{\beta} = 0 \qquad (\lambda_m \neq \lambda_n)$$

$$= \frac{b^2}{2}\left[J_1(b\lambda_n)\right]^2 \qquad (\lambda_m = \lambda_n)$$

The quantity $\omega_n$ is now defined as $b\lambda_n/b$, and the upper limit b is taken to be very large. The difference between two consecutive values of $\omega_n$ may then be obtained by considering large zeros $(b\lambda_n)$ of $J_o$. Thus, the approximating formula for the nth zero is

$$b\lambda_n = \left(n - \frac{1}{4}\right)\pi \tag{26}$$

Hence, the distance between consecutive zeros approaches $\pi$ and the difference between two consecutive values of $\omega_n$ becomes

$$\delta\omega = \frac{\pi}{b}$$

or

$$\frac{1}{b} = \frac{1}{\pi}\,\delta\omega$$

Now, for large values of $b\lambda_n$,

$$J_1(b\lambda_n) = \frac{\sin\left(b\lambda_n - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi}{2}\,b\lambda_n}}$$

or with the aid of equation (26),

$$J_1(b\lambda_n) = \frac{-(-1)^n}{\sqrt{\frac{\pi}{2}\,b\lambda_n}}$$

It follows then from equation (25) that

$$d_n = \omega\delta\omega \int_0^b F(\beta)\sqrt{\beta}J_o(\omega\sqrt{\beta})d\sqrt{\beta}$$

Introducing this expression for $d_n$ into equation (16) for the complementary function $\phi_{1c}$, replacing the summation by integration, and letting the upper limits go to infinity lead to the following expression:

$$\phi_{1c} = \sqrt{2}\int_0^\infty \frac{K_o(\omega\sqrt{\alpha})J_o(\omega\sqrt{\beta})}{K_1\left(\frac{\omega}{\sqrt{2}}\right)} d\omega \int_0^\infty F(t^2)J_o(\omega t)t\ dt \tag{27}$$

Inserting this expression for the complementary function into equation (23) then yields the exact form for the second term in the Janzen-Rayleigh method for the calculation of subsonic flow past a paraboloid or revolution. Equation (27) can be verified by means of the boundary condition

$$\left(\frac{\partial\phi_{1c}}{\partial\alpha}\right)_{\alpha=\frac{1}{2}} = F(\beta)$$

and the recurrence relation

$$\left[\frac{\partial K_o(\omega\sqrt{\alpha})}{\partial\alpha}\right]_{\alpha=\frac{1}{2}} = -\frac{\omega}{\sqrt{2}} K_1\left(\frac{\omega}{\sqrt{2}}\right)$$

Thus the well-known Fourier-Bessel integral (ref. 3) is obtained:

$$F(\beta) = \int_0^\infty \omega\ d\omega \int_0^\infty F(t^2)J_o(\omega t)J_o(\omega\sqrt{\beta})t\ dt$$

## NUMERICAL CONSIDERATIONS

Attempts to evaluate the right-hand side of equation (27), or even to reduce it to a single infinite integral, have thus far proved fruitless. It appears certain that, in order to obtain numerical results, equation (27) for $\phi_{1c}$ or its derivatives with regard to $\alpha$ and $\beta$ must be calculated directly. This calculation is not done in the present paper. Rather, in preparation for such numerical computations it is necessary to evaluate the integral appearing in the expression (eq. (24)) for $F(\beta)$; thus,

$$I(\beta) = \int_0^{\frac{1-2\beta}{1+2\beta}} \frac{\log(1 + \mu) - \log 2}{1 - \mu} \, d\mu \qquad (0 \leqq \beta \leqq \infty)$$

Then, the following expansions are considered:

$$\log(1 + \mu) = -\sum_{n=1}^{\infty} (-1)^n \frac{\mu^n}{n} \qquad (-1 < \mu \gtreqless 1)$$

and

$$\log 2 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

It follows that

$$\frac{\log(1 + \mu) - \log 2}{1 - \mu} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{m=1}^{n} \mu^{m-1}$$

Hence

$$I(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{m=1}^{n} \frac{\mu^m}{m} \qquad (-1 \leqq \mu \leqq 1) \qquad (28)$$

Note that the convergence has been improved by the act of integration to include $\mu = -1$. Thus,

$$\lim_{\substack{\beta \longrightarrow 0 \\ \text{or} \\ \mu \longrightarrow 1}} I(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} S_n$$

where

$$S_n = \sum_{m=1}^{n} \frac{1}{m}$$

Now

$$\log^2 2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} S_n - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Therefore,

$$\lim_{\beta \longrightarrow 0} I(\beta) = \frac{1}{2} \log^2 2 - \frac{\pi^2}{12} \approx -0.58224 \tag{29}$$

For $\beta \longrightarrow \infty$ or $\mu \longrightarrow -1$, equation (28) gives

$$\lim_{\beta \longrightarrow \infty} I(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{m=1}^{n} \frac{(-1)^m}{m}$$

or after rearrangement of terms on the right-hand side

$$\lim_{\beta \longrightarrow \infty} I(\beta) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} S_n + \sum_{n=1}^{\infty} \frac{S_n}{2n(2n + 1)}$$

$$= -\lim_{\beta \longrightarrow 0} I(\beta) + \sum_{n=1}^{\infty} \frac{S_n}{2n(2n + 1)}$$

The series on the right-hand side can be rendered more rapidly convergent by repeated application of Kummer's transformation (ref. 4); thus,

$$\lim_{\beta \longrightarrow \infty} I(\beta) = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2 + \frac{19}{60} + \frac{9}{4} \sum_{n=1}^{\infty} \frac{1}{(n + 1)(2n + 3)(2n + 5)} +$$

$$\frac{15}{2} \sum_{n=1}^{\infty} \frac{S_n}{n(2n + 1)(2n + 3)(2n + 5)}$$

$$\approx 1.060$$

Finally, for $0 < \beta < \infty$ or $\mu^2 < 1$,

$$I(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{m=1}^{n} \frac{\mu^m}{m}$$

$$= \log 2 \log(1 - \mu) - \sum_{n=1}^{\infty} \frac{\mu^{n+1}}{n + 1} \sum_{m=1}^{n} \frac{(-1)^m}{m}$$

$$= \log 2 \log(1 - \mu) - \frac{1}{2} \log(1 + \mu)\log(1 - \mu) + \sum_{n=1}^{\infty} \frac{\mu^{2n+1}}{2n + 1} t_{2n}$$

where

$$t_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ . \ . \ . - \frac{1}{2n} = S_{2n} - S_n$$

Thus,

$$I(\beta) = \frac{1}{2} \log \frac{4\beta}{1 + 2\beta} \log 2(1 + 2\beta) + \sum_{n=1}^{\infty} \frac{t_{2n}}{2n + 1}\left(\frac{1 - 2\beta}{1 + 2\beta}\right)^{2n+1} \qquad (0 < \beta < \infty) \tag{30}$$

With the aid of equations (29) and (30) the function $F(\beta)$ can be computed for all finite values of $\beta$. For $0 < \beta < \infty$, the expression for $F(\beta)$ in equation (24) becomes

$$F(\beta) = \frac{1}{(1 + 2\beta)^3}\left\{\frac{3}{2}(1 - 2\beta)\left[-1 + \log(1 + 2\beta)\right] + (1 + 2\beta)\log 2\right\} +$$
$$\frac{1 - 4\beta}{(1 + 2\beta)^4}\left[\frac{1}{2} \log 2\beta \log 2(1 + 2\beta) + \sum_{n=1}^{\infty} \frac{t_{2n}}{2n + 1}\left(\frac{1 - 2\beta}{1 + 2\beta}\right)^{2n+1}\right]$$

Table I lists values of $F(\beta)$ and $I(\beta)$ for the range $0 \lesseqgtr \beta \lesseqgtr 9$, and figures 2 and 3 show the graphs of each. For the later purpose of calculating the velocity distribution in the neighborhood of the stagnation point and also along the axis of symmetry of the flow, general expressions are presented in the following section.

## CALCULATION OF FLUID VELOCITY AT THE SOLID BOUNDARY AND ALONG THE AXIS OF SYMMETRY OF FLOW

The fluid velocity at a point on the body is given by

$$q_s = \frac{\partial\phi}{\partial s_\eta} = \frac{1}{J}\frac{\partial\phi}{\partial\eta}$$

where $ds_\eta$ is the element of arc in the direction in which $\eta$ increases and

$$J^2 = \frac{dz}{d\zeta}\frac{d\bar{z}}{d\bar{\zeta}}$$

Then, in terms of the variable $\beta = \eta^2$,

$$q_s = \sqrt{\frac{2\beta}{1 + 2\beta}}\left(\frac{\partial\phi}{\partial\beta}\right)_{\alpha=\frac{1}{2}} \tag{31}$$

With the aid of equations (12) and (23), equation (31) yields the following expression for the fluid speed along the solid boundary for the range $0 \gtreqless \beta < 1$:

$$q_{sM_\infty} = q_{so}\left(1 + M_\infty^2\left\{\frac{1 - \beta}{(1 + 2\beta)^4}\left[\log\frac{2}{1 + 2\beta}\log\frac{1}{2(1 + 2\beta)} + 2I(\beta)\right] - \frac{1 - 2\beta}{(1 + 2\beta)^4}\frac{\log(1 + 2\beta)}{4\beta} - \frac{(1 - 2\beta)(1 + 4\beta)}{2(1 + 2\beta)^4}\log\frac{2}{1 + 2\beta} + \frac{1}{2(1 + 2\beta)^2} - \frac{6 - 5\log 2}{2(1 + 2\beta)^3} - \frac{1}{(1 + 2\beta)^4}\log 2 + \left(\frac{\partial\phi_{1c}}{\partial\beta}\right)_{\alpha=\frac{1}{2}}\right\}\right) \tag{32}$$

where $q_{so} = \sqrt{\frac{2\beta}{1 + 2\beta}}$ is the magnitude of the fluid velocity at the surface in incompressible flow and $\phi_{1c}$ is given by equation (27).

The fluid velocity along the axis of symmetry of the flow is given by

$$q_x = \frac{\partial\phi}{\partial x} = \left(\frac{\partial\phi}{\partial\alpha}\right)_{\beta=0}$$

Then with the aid of equations (12) and (23),

$$q_{xM_\infty} = q_{xo}\left\{1 + \frac{M_\infty^2}{4x(2x - 1)}\left[1 - \log 8x + \frac{1}{x}\left(1 + \frac{\pi^2}{12}\right) + 8x^2\left(\frac{\partial\phi_{1c}}{\partial\alpha}\right)_{\beta=0}\right]\right\} \tag{33}$$

where $q_{xo} = -1 + \frac{1}{2x}$ is the velocity of the fluid along the axis of symmetry in incompressible flow. Equations (32) and (33) have the form of correction factors by which $q_{so}$ and $q_{xo}$ are multiplied in order to take into account the effect of compressibility. It may be desirable to utilize the arc length $s_\beta$ of the parabolic meridian profile as the reference variable rather than $\beta$. The arc length $s_\beta$, with the radius of curvature at the nose as unit of length, is given by

$$s_\beta = \frac{1}{2}\left[\sqrt{2\beta(1 + 2\beta)} + \log\left(\sqrt{2\beta} + \sqrt{1 + 2\beta}\right)\right] \tag{34a}$$

or

$$s_\beta = \frac{\sin\frac{\theta}{2}}{1 + \cos\theta} + \frac{1}{2}\log\frac{1 + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \tag{34b}$$

For purposes of comparison, consider the two-dimensional case of uniform subsonic flow past a parabolic cylinder. The following formulas, corresponding to equations (32) and (33), are easily obtained from reference 5:

$$q_{sM_\infty} = q_{so}\left\{1 - \frac{1}{4}M_\infty^2\cos^2\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\left(\frac{\theta\cos\theta}{\sin\frac{\theta}{2}} - 4\cos\frac{\theta}{2}\log 2\cos\frac{\theta}{2}\right)\right]\right\} \tag{35}$$

and

$$q_{xM_\infty} = q_{xo}\left\{1 - \frac{1}{2}M_\infty^2\left[\frac{1}{\sqrt{2x}} - \frac{\sqrt{2}}{4x\left(\sqrt{x} - \sqrt{2}\right)} + \frac{\log\frac{1}{2}x}{4x\left(\sqrt{x} - \sqrt{2}\right)^2}\right]\right\} \tag{36}$$

where $q_{so} = \sin\frac{\theta}{2}$ and $q_{xo} = -1 + \frac{1}{\sqrt{2x}}$.

## CONCLUDING REMARKS

In conclusion, the present paper provides an additional example to the sparse literature on subsonic axisymmetrical flow. The attempt has been made to choose as solid boundary a shape which does not require involved and cumbersome analysis but, at the same time, which is of interest to both theoretical and applied aerodynamicists. Attention is particularly directed to the double infinite integral of equation (27). Such integrals, involving Bessel functions under the integral signs, are not only of great interest to the pure mathematician but are also of extreme importance in many branches of mathematical physics. The addition of another one to the large number of such integrals which have already been evaluated can generally be counted upon to aid in the solution of many problems in varied fields.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., March 14, 1956.

## REFERENCES

1. Longhorn, A. L.: Subsonic Compressible Flow Past Bluff Bodies. Aero. Quarterly, vol. V, pt. 2, July 1954, pp. 144-162.

2. Munk, Max M.: Fluid Mechanics, Pt. II. Ellipsoid With Three Unequal Axes. Vol. 1 of Aerodynamic Theory, div. C, ch. VIII, sec. 4, W. F. Durand, ed., Julius Springer (Berlin), 1934, p. 298.

3. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second ed., The Macmillan Co., 1944, p. 453.

4. Adams, Edwin P., and Hippisley, R. L.: Smithsonian Mathematical Formulae and Tables of Elliptic Functions. Second reprint, Smithsonian Misc. Coll., vol. 74, no. 1, 1947, p. 114.

5. Kaplan, Carl: On the Small-Disturbance Iteration Method for the Flow of a Compressible Fluid With Application to a Parabolic Cylinder. NACA TN 3318, 1955.

TABLE I

TABULATED VALUES OF THE FUNCTIONS $I(\beta)$ AND $F(\beta)$

| $\beta$ | $I(\beta)$ | $F(\beta)$ | $\beta$ | $I(\beta)$ | $F(\beta)$ |
|---|---|---|---|---|---|
| 0 | -0.58224 | -1.62932 | 0.875 | 0.20199 | 0.07033 |
| .02083 | -.54184 | -1.18932 | 1.000 | .25103 | .04879 |
| .04167 | -.50378 | -.85251 | 1.200 | .31729 | .02452 |
| .06250 | -.46788 | -.59387 | 1.400 | .37215 | .00895 |
| .08333 | -.43393 | -.39490 | 1.600 | .41845 | -.00105 |
| .10417 | -.40176 | -.24175 | 1.700 | .43900 | -.00461 |
| .12500 | -.36666 | -.12398 | 1.800 | .45812 | -.00746 |
| .14583 | -.34222 | -.03368 | 1.900 | .47591 | -.00974 |
| .16667 | -.31459 | .03524 | 2.000 | .49255 | -.01155 |
| .18750 | -.28825 | .08742 | 2.500 | .56184 | -.01612 |
| .20833 | -.26309 | .12650 | 3.000 | .61442 | -.01693 |
| .22917 | -.23905 | .15527 | 3.500 | .65589 | -.01633 |
| .25000 | -.21602 | .17595 | 4.000 | .68955 | -.01523 |
| .31250 | -.15249 | .20490 | 5.000 | .74108 | -.01283 |
| .37500 | -.09606 | .20510 | 6.000 | .77160 | -.01074 |
| .43750 | -.04555 | .19195 | 7.000 | .80804 | -.00904 |
| .50000 | 0 | .17329 | 8.000 | .83123 | -.00771 |
| .56250 | .04132 | .15309 | 9.000 | .85016 | -.00664 |
| .62500 | .07901 | .13333 | ∞ | 1.06000 | 0 |
| .75000 | .14535 | .09829 | | | |

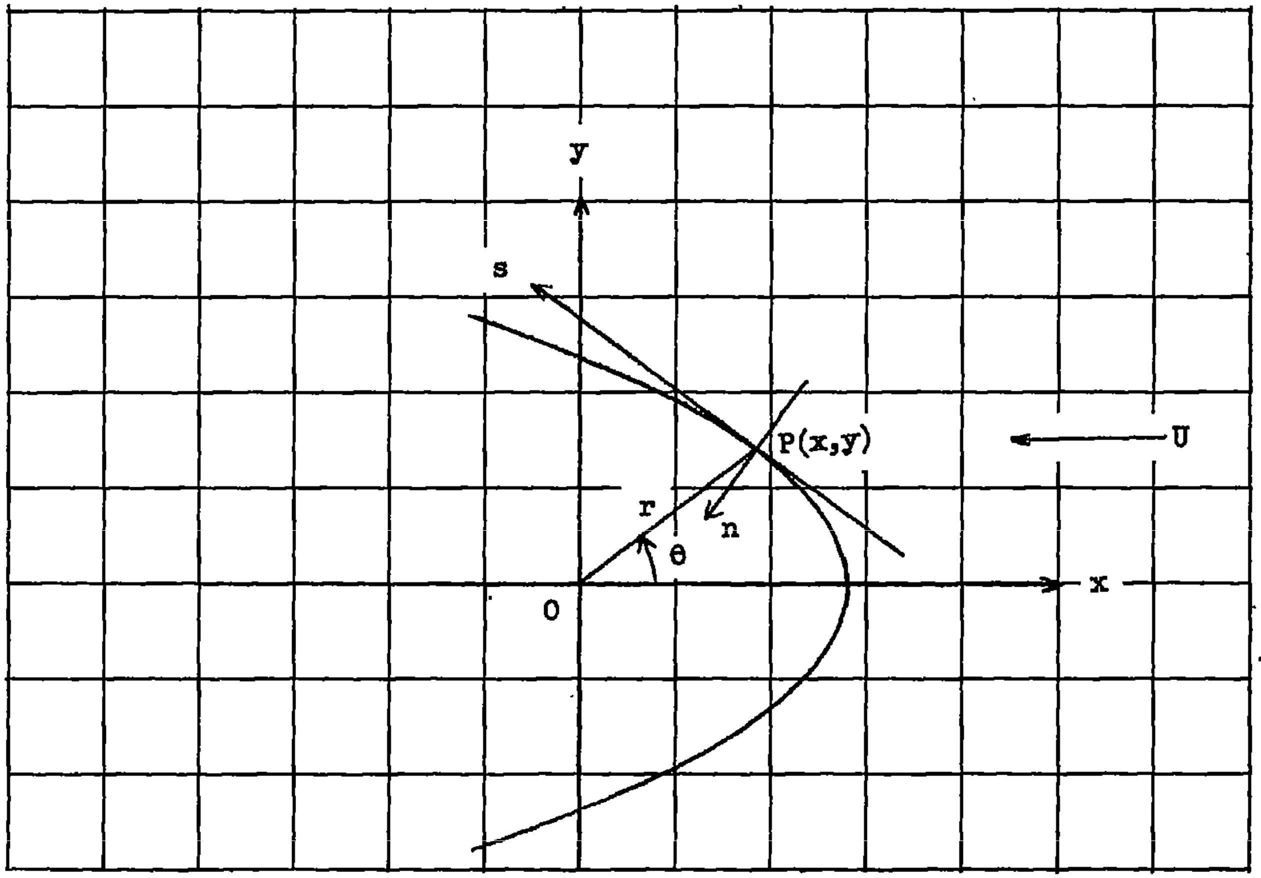


Figure 1.- Profile of paraboloid of revolution in meridian xy-plane.

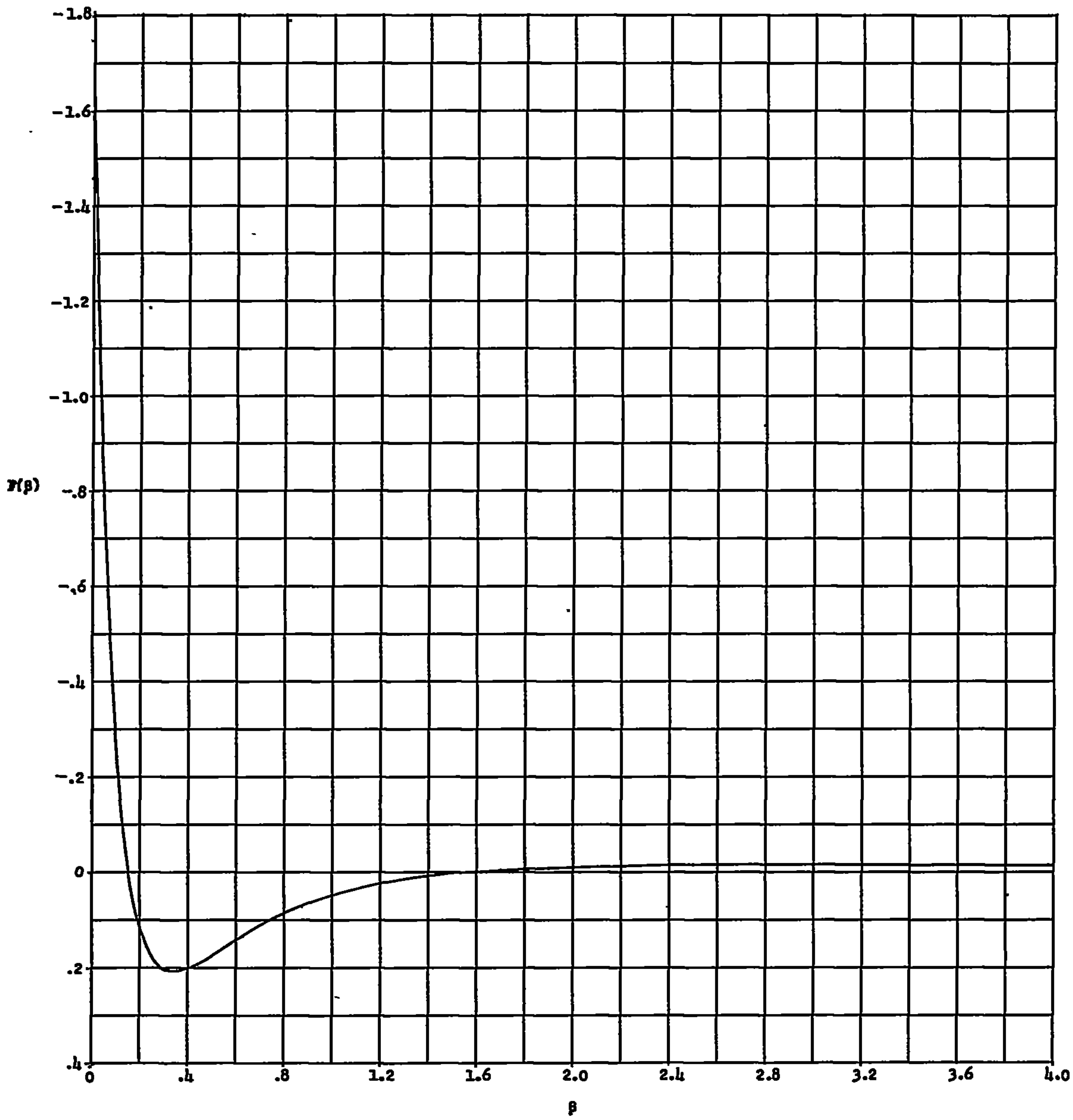


Figure 2.- The boundary function $F(\beta)$ plotted against $\beta$.

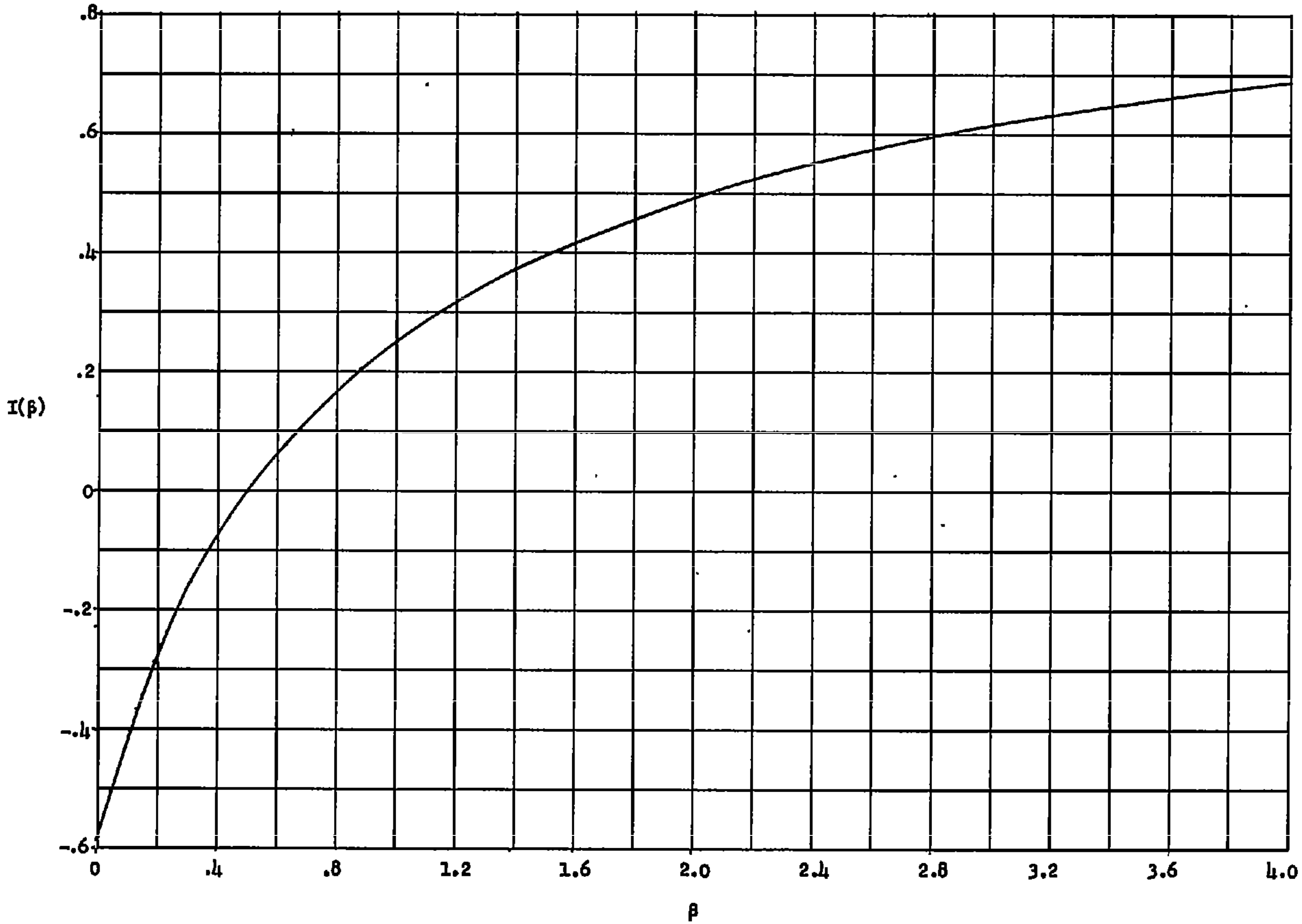


Figure 3.- The integral $I(\beta)$ plotted against $\beta$.